Feedback control of canards

Joseph Durham\textsuperscript{a)} and Jeff Moehlis\textsuperscript{b)}
Department of Mechanical Engineering, University of California, Santa Barbara, California 93106, USA
(Received 9 January 2007; accepted 11 October 2007; published online 27 March 2008)

We present a control mechanism for tuning a fast-slow dynamical system undergoing a supercritical Hopf bifurcation to be in the canard regime, the tiny parameter window between small and large periodic behavior. Our control strategy uses continuous feedback control via a slow control variable to cause the system to drift on average toward canard orbits. We apply this to tune the FitzHugh-Nagumo model to produce maximal canard orbits. When the controller is improperly configured, periodic or chaotic mixed-mode oscillations are found. We also investigate the effects of noise on this control mechanism. Finally, we demonstrate that a sensor tuned in this way to operate near the canard regime can detect tiny changes in system parameters. © 2008 American Institute of Physics. [DOI: 10.1063/1.2804554]

Canards are special periodic orbits that are associated with a dramatic change in amplitude and period due to a very small change in a parameter. Since canards typically exist only for very small regions of parameter space, they are extremely difficult to observe experimentally. In this paper we present a continuous feedback control mechanism that uses only the instantaneous position of the system in phase space to tune the system to be in the canard regime. This involves controlling a slow variable to drift on average toward the canard parameter region, and is inspired by the dynamics of mixed-mode oscillations. A system controlled in this way could serve as a sensor that can detect extremely small changes in system parameters.

I. INTRODUCTION

Canards are periodic orbits for which the trajectory follows both attracting and repelling slow manifolds. They are associated with a dramatic change in amplitude and period over a very narrow interval of a parameter. Canards may be present in singularly perturbed systems of ordinary differential equations: a common scenario in which they arise is that a “small” stable periodic orbit is born in a supercritical Hopf bifurcation and rapidly changes to a “large” relaxation oscillation periodic orbit as a parameter is varied. Canards are the intermediate periodic orbits between the small and large orbits. The shape of these periodic orbits in phase space can resemble a duck; hence the name “canard,” the French word for duck. Canards were first found in studies of the van der Pol system,\textsuperscript{1–3} and have since been found and analyzed to varying degrees for a variety of chemical, biological, and other systems.\textsuperscript{4–22} Because canards typically only exist for very small regions of parameter space, they are extremely difficult to observe experimentally.

In this paper we present a control mechanism that tunes a system to be in the canard regime. This is a continuous feedback control law that uses only the instantaneous position of the system in phase space, and is conceptually similar to one approach used for tuning a system to be at a Hopf bifurcation.\textsuperscript{23} Our control mechanism is inspired by the relationship of canards to mixed-mode oscillations (MMOs), which are solutions consisting of sequences of small and large orbits in phase space, as determined by whether the traced orbits are smaller or larger than the corresponding canard solutions. MMOs have been found and analyzed for various systems\textsuperscript{24–29} (also see the other articles in this issue). MMO occur, for example, for fast-slow dynamical systems when a variable on average drifts toward and then across a transition from its present state (respectively, tracing a large or small orbit) to a different state (respectively, tracing a small or large orbit). Most commonly, such transitions occur periodically, giving MMOs that can be characterized by the repeating sequence in which the small and large orbits occur, although chaotic MMOs can also occur. Our control mechanism involves a slow variable that similarly drifts toward the canard transition. However, unlike MMOs, by tuning the dynamics of the control variable appropriately, it is possible for the controlled system to converge to the canard behavior, rather than repeatedly switching between a large and small orbit.

A system tuned to be at or near a point in parameter space for which a bifurcation or canard transition occurs can be used to sense parameter changes: one type of behavior indicates the parameter changed in one direction, while another type of behavior indicates the parameter changed in the other direction. For example, suppose a system is tuned to be at a supercritical Hopf bifurcation, as in Fig. 1(a). If the parameter decreases a stable fixed point will be reached, while if the parameter increases a stable periodic orbit will be reached. However, the size of the periodic orbit shrinks to zero as the Hopf bifurcation is approached; if it is difficult to distinguish a fixed point from a small periodic orbit, such a system would have trouble detecting small parameter changes. On the other hand, suppose that a system is tuned to be at a subcritical Hopf bifurcation, with the periodic orbit gaining stability in a saddle-node bifurcation, as in Fig. 1(b).
If the parameter increases even a small amount, a large periodic orbit will be reached, which might be easily distinguished from a stable fixed point. However, such a system would not easily detect a subsequent small decrease in the parameter: hysteresis makes the system stay on the stable periodic orbit branch. In order to “reset” such a sensor, it would not easily detect a subsequent small decrease in the parameter: hysteresis makes the system stay on the stable periodic orbit branch. In order to “reset” such a sensor, it would be necessary to decrease the parameter by a substantial amount (past the saddlenode bifurcation), then retime the system to be at the subcritical Hopf bifurcation. In contrast, consider a system that is tuned to be at a canard transition near a supercritical Hopf bifurcation, as in Fig. 1(c). The presence of a large periodic orbit indicates a positive change in the parameter, while the presence of a small periodic orbit indicates a negative change in the parameter. Note that, because of the nature of the canard transition, this will be true even for very small changes in the parameters. If it is relatively easy to distinguish a large from a small periodic orbit, such a sensor could detect extremely small parameter changes while avoiding issues with hysteresis. These properties make this final canard scenario a good candidate for sensing tasks. In our present application, we envision using a slow control variable to tune the system to be in the canard regime, then using the system to detect a small change in a system parameter.

In Sec. II, we discuss the presence of canards in the FitzHugh-Nagumo (FHN) model, a prototypical model for neural dynamics that will serve as the example throughout this paper. In Sec. III, we describe the control mechanism that tunes our system to be in the canard regime. Next, in Sec. IV, we determine how well the control works for different parameters in the control law. This includes the result that for certain parameters, the control law leads to MMOs. We also consider this control mechanism for this system subjected to white noise. In Sec. V, we demonstrate how a sensor tuned to operate near the canard regime can detect tiny changes in system parameters. We give concluding thoughts in Sec. VI. While we focus on the FitzHugh-Nagumo equations, we expect that the mechanism that we describe will work for other appropriate systems, provided appropriate tuning of the parameters in the control law is done.

II. CANARDS

The system we consider is the FitzHugh-Nagumo model of neuron spiking behavior. The dynamics are described by the differential equations

\[ \dot{v} = -w - v(v - 1)(v - a) + I = f(v, w; I), \]

\[ \dot{w} = \epsilon(v - \gamma w) = \epsilon g(v, w). \]

Here, \( \epsilon \ll 1 \) is a time-scale separation parameter, and \( v \) and \( w \) refer to voltage and recovery variables, respectively. Following Brøns, we set \( a=0.1, \gamma=1, \) and \( \epsilon=0.008 \). The parameter \( I \) represents an external current applied to the model, and in this section we treat it as a bifurcation parameter. As shown in Fig. 2, the FHN model with these parameters undergoes a supercritical Hopf bifurcation around \( I=0.0553 \) and shortly thereafter, the amplitude and shape of the stable periodic orbit change dramatically over a very narrow range of \( I \). This narrow range of \( I \) is the canard region: a sample of canard periodic orbits is shown in Fig. 3. For this paper we will consider orbits to be small periodic orbits if they are smaller than orbit (a) and large periodic orbits if they are larger than orbit (e) in Fig. 3.

FIG. 1. Bifurcation diagrams showing fixed point (f.p.) and periodic orbit (p.o.) branches for (a) supercritical Hopf bifurcation, (b) subcritical Hopf bifurcation, and (c) canard transition. Solid (dashed) lines indicate stable (unstable) solutions, while the arrows point to the location of the transition. The parameter changes along the horizontal axis, and the vertical axis is a measure of the size of the periodic orbit.
The canard phenomenon can be understood as follows. The system has two nullclines: a cubic $v$-nullcline, where $f(v,w)=0$, and a linear $w$-nullcline, where $g(v,w)=0$. If $\epsilon$ is set equal to zero, then $\dot{w}=0$ and the $v$-nullcline is a curve of fixed points and is normally hyperbolic on the pieces for which its slope is bounded away from zero; i.e., away from $O$. The $w$-nullcline is a slow manifold, with stable foliation, within particular utility, Eqs. The manifolds near the local minimum of the $v$-nullcline. This distance of the right part of the $v$-nullcline. A trajectory follows $M_s$, and after passing near the local minimum of the $v$-nullcline, either (a) returns quickly to a neighborhood of $M_s$, or (b) undergoes a large excursion before returning to a neighborhood of $M_s$. 

FIG. 3. (Color online) Stable periodic orbit evolution over a small range of parameter $I$ in the FHN model: (a) $I=0.056$ 838, (b) $I=0.056$ 838 45, (c) $I=0.056$ 838 84, (d) $I=0.056$ 838 58, (e) $I=0.056$ 84. Here, and in later phase plane figures, the dashed line is the $v$-nullcline, where $f(v,w)=0$.

Hopf bifurcation point ($I_H$ in our notation below) and canard point ($I_C$ in our notation below), respectively. Since functions $f$ and $g$ in Eqs. (1) and (2) have no dependence on $\epsilon$, Brøns’ equations in our notation simplify to

$$a_1 = -\frac{1}{16} (f_w^2 g_w g_{ww} f_{v w w} + g_w^2 f_{v v w} f_{v w w}),$$

$$\delta = \frac{g_w g_{ww} f_{v w}}{g_{v w} g_w},$$

$$I_H(\epsilon) = I_{base} - \frac{g_w g_{ww} f_{v w}}{g_{v w} g_w} \epsilon + O(\epsilon^2),$$

$$I_C(\epsilon) = I_H(\epsilon) - \frac{8 a_1}{f_w^2} \epsilon + O(\epsilon^2).$$

Here, $I_{base}$ is the parameter value at which the $w$-nullcline intersects the $v$-nullcline at its local minimum, and the derivatives are evaluated at this intersection in phase space. For our system (1) and (2), this intersection occurs at

$$v_i = (1 + \sqrt{1 - a + a^2})/3 \approx 0.048 687,$$

$$w_i = v_i/\gamma \approx 0.048 687,$$

$$I_{base} = v_i - 2 v_i^2 + v_i w_i - 2 v_i^2 w_i + v_i^3 w_i - \gamma - 2 w_i \approx 0.051 064,$$

and this theory predicts that the Hopf bifurcation point and canard point will occur at $I_H \approx 0.0553$ and $I_C \approx 0.0566$, respectively. Both of these results are accurate to first order in $\epsilon$, and match the numerical results shown in Fig. 2 to that order.

III. CONTROL METHOD

Our goal is to design a control mechanism that steers the FHN model to the canard regime, without precise foreknowledge of where this occurs in parameter space. Since $I$ is a natural bifurcation parameter in Eqs. (1) and (2), it makes a natural control variable as well. We will choose the control law so that $I$ evolves slowly enough that the behavior for an instantaneous (but slowly changing) value of $I$ can be well approximated by that predicted from Fig. 2 for constant $I$. If the dynamics are (approximately) those of a large periodic orbit, $I$ should then be decreased, and if the dynamics are (approximately) those of a small periodic orbit, $I$ should be

FIG. 4. The two generic situations for the relative positions of the slow manifolds $M_s$ and $M_U$ near the local minimum of the $v$-nullcline. A trajectory follows $M_s$, and after passing near the local minimum of the $v$-nullcline, either (a) returns quickly to a neighborhood of $M_s$, or (b) undergoes a large excursion before returning to a neighborhood of $M_s$. 

Author complimentary copy. Redistribution subject to AIP license or copyright, see http://cha.aip.org/cha/copyright.jsp
increased. If the changes in $I$ cause it to drift back and forth across the canard region, this strategy would produce MMOs. However, when properly tuned, as described below, it will converge to the intermediate canard orbits.

We choose to use continuous feedback control based on the position of the system in phase space, similar to the approach of Moreau and Sontag for tuning to a Hopf bifurcation. The local minimum of the $v$-nullcline is the base point for our measurements and we construct a control circle around this point to determine whether trajectories are instantaneously on a small or large orbit. Specifically, we will assume that a trajectory outside the control circle is (approximately) on a large periodic orbit, and $I$ should be decreased. On the other hand, a trajectory inside the control circle is assumed to (approximately) be on a small periodic orbit, and $I$ should be increased. While this identification is not always correct (for example, large periodic orbits spend some time within the control circle), we show below that under proper tuning, $I$ will cycle over a small range, with a corresponding canardlike trajectory that balances the effects of sometimes being inside and sometimes outside the control circle.

To include this control strategy in the FHN model, we augment Eqs. (1) and (2) with the following differential equation for $I$:

$$
\dot{I} = c(r_0 - r).
$$

The new variable $r = \sqrt{(u-v)^2 + (w-w_i)^2}$ is the instantaneous Euclidean distance from the local minimum of the $v$-nullcline. The parameters $c$ and $r_0$ determine the control strength and radius of the control circle, respectively, and will be tuned as described in the next section to produce a “good” canard. This control strategy is memoryless, as it depends only on the instantaneous position in phase space, and also does not require foreknowledge of the parameter values for which canards exist. That being said, it is only expected to work for fast-slow systems near a supercritical Hopf bifurcation, where the system must begin in an oscillatory region of parameter space but can start with the control variable on either side of the canard point. This control strategy also requires approximate knowledge of the size of periodic orbits on either side of the canard point to pick an “good” canard. This control strategy is memoryless, as it will be tuned as described in the next section to produce a

With these considerations in mind, our distance measurement begins when the trajectory passes the local minimum of the $v$-nullcline. We consider the trajectory to have departed the neighborhood of $M_U$ when its slope differs by 0.09 from that of the $v$-nullcline with the same value of $v$. This value, which is an order of magnitude larger than $\epsilon$, was chosen so that the trajectory with the longest measurement has a large change in slope just before the local maximum of the $v$-nullcline. Fig. 5 shows several trajectories and how our method classifies their distance. We emphasize that this distance measure is only used for diagnostic purposes; it is not used in the control law itself.

IV. TUNING TO THE CANARD REGIME

To compare the effectiveness of this control strategy for various values of $c$ and $r_0$, we would like to measure the distance over which the trajectory remains in the neighborhood of the slow manifold $M_U$. This manifold is within $O(\epsilon)$ of the middle portion of the $v$-nullcline. This implies that the slope of $M_U$ must be close to that of $v$-nullcline. When a trajectory departs from a neighborhood of $M_U$, it does so abruptly, making a sharp turn with a large change in slope.

The results of a two-parameter study of $c$ and $r_0$ using this distance measure are shown in Fig. 6, where we average over multiple visits near $M_U$ to account for the possibility of MMOs (see below). Using other values for this slope difference threshold results in a slightly different specific largest canard, but the results are very similar. As $r_0$ increases from 0.15, the distance over which trajectories remain in the neighborhood of $M_U$ generally increases until reaching its peak around $r_0=0.234$. The longest canard orbit in our study occurs for $c=10^{-8}$ and $r_0=0.234$, and is shown in Fig. 7; we will refer to this as the maximal canard. Just above this value of $r_0$, the length of trajectories drops sharply as the trajectories turn off a shorter distance up $M_U$. As the two-parameter study shows, using a smaller value of $c$ produces longer tracking of $M_U$ and thus more canardlike shapes. Using values of $c$ less than $10^{-8}$ does not significantly improve the distance measure.

The value of $c$ must be smaller than $10^{-4}$ to tune the system to the canard region. As demonstrated in Fig. 3, the size of the canard region in $I$ is smaller than $10^{-5}$, with maximal canards in a range of $I$ several orders of magnitude smaller. Our choice of memoryless, continuous control also mandates very small corrections. This size constraint on $c$
effectively creates three time scales for the system (1), (2), and (10), as $1 \gg \epsilon \gg c$. Over the maximal canard trajectory, the control variable $I$ is never stationary but enters into a repeating cycle, as shown in Fig. 8. As the trajectory passes near the local minimum of the $v$-nullcline, it is in the center of the control circle and $I$ increases most rapidly. The trajectory then passes out of the circle on its way up $M_U$, and $I$ starts to decrease more rapidly as the orbit moves through the canard’s “head.” On its return to $M_S$, the trajectory briefly passes through the top of the control circle, resulting in the short reversal in Fig. 8. As this occurs away from the $v$-nullcline, the trajectory is moving quite rapidly, keeping the reversal small.

While the canard trajectory shown in Fig. 7 traces a single orbit each time around, this is not always the case. For $c=10^{-8}$, we also found MMOs with one large and one small orbit, as shown in Fig. 9. These MMOs occur when the control strategy overcorrects for the value of $I$. When the trajectory departs the local minimum of the $v$-nullcline headed for a large orbit, it spends a substantial amount of time outside the control circle, which lowers the value of $I$. When the trajectory re-enters the control circle, $I$ begins to increase again. If the control circle is too small and/or the control strength $c$ too large, then when the trajectory departs again it will have overcorrected the value of $I$, leading to a small orbit. The trajectory then spends a substantial amount of time inside the control circle, which increases the value of $I$, and can lead to another large orbit.

FIG. 6. Contour plot of the average distance the trajectories remain in the neighborhood of $M_U$ after transients have died out. Average is taken over at least 40 successive visits near $M_U$.

FIG. 7. (Color online) This canard trajectory, produced using control with $c=10^{-8}$ and $r_0=0.234$, has the longest distance measure along $M_U$. The axes are not square, so the dot-dashed control circle appears elliptical.

FIG. 8. Evolution of $I$ for the trajectory in Fig. 7.

FIG. 9. Bifurcation diagram for $c=10^{-8}$ showing peak values of $v$, generated by adiabatically increasing the value of $r_0$, omitting transients. There is a period-2 bubble corresponding to a MMO with one small and one large orbit as $r$ is swept from 0.18 to 0.25. The maximal canard has $v_{peak} = 0.65$. 
When $c$ is increased, the window of MMOs expands. Figure 10 shows bifurcation diagrams for $c = 2 \times 10^{-8}$, $5 \times 10^{-8}$, and $1 \times 10^{-7}$. For all three of these, the period-doubling “bubble” in Fig. 9 expands into cascades of period-doubling bifurcations, that leads to chaotic MMOs. As $c$ increases, the width of the region with complex behavior broadens as the propensity for overcorrection in $I$ increases. The chaotic region is broken up by windows of MMO periodic orbits, with the number of MMO windows increasing with $c$. Each of these windows corresponds to a different type of MMO, beginning with $1^n$ orbits ($L^s$ means $s$ small orbits for every $L$ large orbits) for small values of $r_0$, transitioning through $1^L$ in the middle of the chaotic region, and ending as $n^L$ orbits, as can be seen best in Fig. 10(c). The same chaotic MMO bifurcation structure has been observed experimentally for the Belousov-Zhabotinski reaction, as well as an electrochemical system. These results are also very reminiscent of results of Petrov et al. Other chaotic MMOs reported recently have a more classical period-doubling cascade bifurcation structure, and can be interestingly interpreted as spikes triggered by a chaotic background.

Figure 11 shows the chaotic trajectory for $c = 10^{-7}$ and $r_0 = 0.17$ with the associated time series for $I$. In Fig. 12, the
(v, w)-phase plane is expanded with a third dimension for the
control variable I, creating a three-dimensional view of the
chaotic trajectory in Fig. 11. The v-nullcline is expanded into
the I-dimension to form a two-dimensional folded surface S,
with the line of local minima of the v-nullcline now referred
to as the fold-line F. This chaotic behavior is not the product
of integration error or other noise, as a map from the max
value of v from one orbit to the next is distinctly one dimen-
sional, as shown in Fig. 13.

We also studied the effects of moving the control circle
so it was not centered on the local minimum of the
v-nullcline. The control strategy still functions when the
circle is displaced by less than half of r0. These results indi-
cate that the control strategy is effective without precise po-
 tioning of the circle, although the specific end canard or
MMO behavior does change when the circle moves. How-
ever, if the circle is displaced so it no longer contains the
local minimum of the v-nullcline, the controller cannot work.

Our control method is robust to large, but infrequent,
changes in system properties. For the FHN model, we use
steps in γ to simulate these sudden changes. As shown in
Fig. 14(a), the control method is capable of responding to
these changes and locating the new canard region. The time
it takes the system to reach the canard window depends on
the value of c, with larger values locating the canard window
more quickly and smaller values finding it more precisely. To
reach the canard region both quickly and precisely, we de-
veloped a strategy for adjusting c depending on the past his-
tory of I. Essentially, if I has settled in and keeps oscillating
over the same region, c is reduced to more accurately deter-
mine the canard window, as shown in Fig. 14(b). If I is
moving in one direction for a sufficiently long time, c is
increased to reduce the time until the new canard window is
acquired. These determinations of the trend of I are made by
examining the behavior of I over the last 20 orbits. If the
difference between the average value of I for first five orbits
and the last five orbits is significantly smaller than the stan-
dard deviation of I over the orbits, we consider I to have
settled in and reduce c to locate the canard window more
precisely. Conversely, if the change in the average value of I
is an order of magnitude greater than the standard deviation,
c is increased to decrease the settling time. This adaptation of
c requires knowledge of past values of I, but greatly im-
proves the settling time for smaller values of c. In addition,
this adaptation mechanism allows for rapid, precise conver-
gence to the canard region from an initial condition.

The high precision required to achieve tuning to a spe-
cific canard orbit raises the question of whether the method
will work in the presence of noise. One potential source of
noise for the FHN neuron model is a noisy external current,
which would directly affect the v equation. Considering
Gaussian white noise, Eqs. (1), (2), and (10) are rewritten

\[ \dot{v} = -w - v(v - 1)(v - a) + I + \sqrt{2D} \eta(t), \quad (11) \]
\[ \dot{w} = \epsilon(v - \gamma w), \quad (12) \]
\[ \dot{I} = c(r_0 - r), \quad (13) \]

where \( \eta(t) \) represents Gaussian delta-correlated noise with
zero mean and unit variance that enters the system continu-
ously. To simulate the response of our controlled FHN model
to this noise, we use a fourth-order Runge-Kutta method
adapted for noise.39

Under this type of noise, our controller is able to ap-
proach I values close to the canard transition, but is unable to
produce repeated canard-shaped orbits. Figure 15 shows
what happens to the maximal canard in Fig. 7 when a small
amount of noise is injected. To achieve a canard shape, the
trajectory must follow \( M_U \). Even small amplitudes of white
noise cause the trajectory to depart from \( M_U \) and the control
logic is simply not set up to offset these local effects.
of canard shapes, Eqs. (11)-(13) produce noisy MMOs even for tiny noise strengths. For our system, larger (smaller) values of noise strength \( D \) move the fork in Fig. 15 lower (higher) on \( M_U \). Note that it is possible to produce similar results to those from previous studies of noise-induced spiking \(^{14,18}\) using our method: choose a small value for \( r_0 \) (say, 0.12) and, in the absence of noise, the control will produce small periodic orbits; with noise, the system will sporadically produce large orbits.

To tune to maximal canard orbits in the presence of noise, perhaps an alternative control method could be developed based on deviation of the orbit from \( M_U \). This would have the potential to overcome continuous noise, but would require either specific prior information about the canard system or an adaptive memory.

While the control circle method cannot produce repeated canard shaped trajectories in the presence of noise, it does tune the system to be close to the canard transition. When the system is operating near the canard regime, it is still sensitive to changes in system properties and can detect them, as we will now show.

V. SENSING

We now suppose that the control method described above is being used to tune the system to be near the canard regime. We will demonstrate with several examples that such a system can be used to rapidly detect a very small change in the system parameter \( \gamma \). This relies on the fact that even rather small perturbations to system properties shift the position of the canard transition so that the system will produce only small or large periodic orbits, which can then be distinguished from the system’s behavior in the canard regime.

First, consider the control parameters \( r_0 = 0.234 \) and \( c = 10^{-8} \), which give the maximal canard trajectory in the sense of maximizing the distance over which the trajectory is close to \( M_U \) (see Fig. 7). When \( \gamma \) is decreased by one part in 100 000, the trajectory immediately begins tracing out a small periodic orbit, which can be easily distinguished from the canard orbit (see Fig. 16). After \( \gamma \) is restored to its original value, the system slowly returns to tracing out the maximal canard trajectory. There are no issues with hysteresis associated with this return.

Even when the system is not tuned to give the maximal canard, it is possible to detect very small changes in \( \gamma \): consider the control parameters \( r_0 = 0.17 \) and \( c = 10^{-7} \), which give the chaotic MMO shown in Figs. 11 and 12. As shown in Fig. 17, when \( \gamma \) is increased by a small amount, the trajectory immediately begins tracing out a large periodic orbit,
In future work, several enhancements to the controller could prove beneficial. Adding an integral term to Eq. (10) could enable the system to more quickly locate the canard orbit when initialized far from the canard region. Adding damping might achieve a similar result and reduce the prevalence of MMOs by shrinking the oscillations in I for larger values of c. Several changes would be necessary to counteract continuous white noise, but a controller that estimated the location of $M_\Gamma$ and reduced deviations away from that manifold might prove successful. It would also be interesting to investigate generalizing this control strategy for higher dimensional systems exhibiting canards, using a control cylinder with axis along the fold-line or a hypersphere.

ACKNOWLEDGMENTS

We thank the referees for thoughtful suggestions for improving this paper. We thank the National Science Foundation (Grant No. NSF-0547606) for supporting this research. J.M. was also supported by an Alfred P. Sloan Research Fellowship in Mathematics.

VI. CONCLUSION

We have demonstrated a novel technique for controlling the FHN model to be in the canard regime. With the addition of a differential equation regulating the parameter I, so that I now acts as a slow control variable, the model self-tunes to give canardlike orbits. Indeed, when properly tuned, our continuous, memoryless method produces repeated maximal canard trajectories. MMOs, including chaotic trajectories, were observed for suboptimal control setups. While our method can relocate the precise canard region when one of the parameters in the FHN model changes, it can only find the general vicinity of the canard region when subjected to continuous white noise. Furthermore, we demonstrated that a sensor tuned with such control could detect tiny changes to the operating parameters of the system without the hysteresis issues associated with operating at a subcritical Hopf bifurcation. We note that this control strategy will not stabilize unstable canards; for example, on a branch of periodic orbits arising from a subcritical Hopf bifurcation.\textsuperscript{21}

which can be easily distinguished from the chaotic MMO. When $\gamma$ returns to its original value, the system evolves back to chaotic MMO behavior.

Finally, this setup can be used to detect small changes in $\gamma$ even when the control parameters do not give the maximal canard, and when small noise is present: see Fig. 18 for results with the same control parameters as in Fig. 17, but with noise added. Simulations (not shown) confirm the expected result that when the noise strength is made larger, the size of changes in $\gamma$ that such a sensor can reliably detect is reduced.

If $\gamma$ remains at its new value, the control, which always remains on, will cause the system to eventually evolve to the canard regime for the new parameter value. The sensor will then be ready to detect a subsequent change to the parameter value. If this is desired, it might be beneficial to use the adaptive method as shown in Fig. 14 to more rapidly converge to the new canard regime.

FIG. 18. (Color online) With $r_0=0.17$, $c=10^{-7}$, and a noise strength $D$ of $10^{-7}$, the system is able to detect a change (here, a decrease) in $\gamma$ of $10^{-3}$. The thick, red lines in the middle of the figure show the behavior of the system for the new value of $\gamma$, to be compared with the dashed lines which correspond to how the time series would have evolved had $\gamma$ not changed.
