Appendix: Derivation of the Fokker-Planck Equation

Let $\{X(t) : t \geq 0\}$ be a one-dimensional stochastic process with $t_1 > t_2 > t_3$. We use $P(X_1, t_1; X_2, t_2)$ to denote the joint probability distribution, i.e., the probability that $X(t_1) = X_1$ and $X(t_2) = X_2$, and $P(X_1, t_1 \mid X_2, t_2)$ to denote the conditional (or transition) probability distribution, i.e., the probability that $X(t_1) = X_1$ given that $X(t_2) = X_2$, defined as $P(X_1, t_1; X_2, t_2) = P(X_1, t_1 \mid X_2, t_2)P(X_2, t_2)$. We will assume $X(t)$ is a Markov process, namely,

$$P(X_1, t_1 \mid X_2, t_2; X_3, t_3) = P(X_1, t_1 \mid X_2, t_2). \quad [1]$$

For any continuous state Markov process, the following Chapman-Kolmogorov equation is satisfied (1, 2):

$$P(X_1, t_1 \mid X_3, t_3) = \int P(X_1, t_1 \mid X_2, t_2)P(X_2, t_2 \mid X_3, t_3)dX_2. \quad [2]$$

In the following, we will also assume $X(t)$ is time homogeneous:

$$P(X_1, t_1 + s; X_2, t_2 + s) = P(X_1, t_1, X_2, t_2), \quad [3]$$

so that X is invariant with respect to a shift in time. For simplicity of notation, we use $P(X_1, t_1 - t_2 \mid X_2) \equiv P(X_1, t_1 \mid X_2, t_2)$.

We will now outline the derivation of the Fokker-Planck equation, a partial differential equation for the time evolution of the transition probability density function. This closely follows the derivation in ref. 3. Consider

$$\int_{-\infty}^{\infty} h(Y) \frac{\partial P(Y, t \mid X)}{\partial t} dY, \quad [4]$$

where $h(Y)$ is any smooth function with compact support. Writing

$$\frac{\partial P(Y, t \mid X)}{\partial t} = \lim_{\Delta t \to 0} \frac{P(Y, t + \Delta t \mid X) - P(Y, t \mid X)}{\Delta t}, \quad [5]$$

and interchanging the limit with the integral, it follows that

$$\int_{-\infty}^{\infty} h(Y) \frac{\partial P(Y, t \mid X)}{\partial t} dY = \lim_{\Delta t \to 0} \int_{-\infty}^{\infty} h(Y) \left[\frac{P(Y, t + \Delta t \mid X) - P(Y, t \mid X)}{\Delta t} \right] dY. \quad [6]$$

Applying the Chapman-Kolmogorov identity (Eq. **2**), the right hand side of Eq. **6** can be written as

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\int_{-\infty}^{\infty} h(Y) \int_{-\infty}^{\infty} P(Y, \Delta t \mid Z)P(Z, t \mid X)dZdY - \int_{-\infty}^{\infty} h(Y)P(Y, t \mid X)dY \right]. \quad [7]$$

Interchanging the limits of integration in the first term of Eq. **7**, letting $Y \to Z$ in the second term, and using the identity $\int_{-\infty}^{\infty} P(Y, \Delta t \mid Z)dY = 1$, we have

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\int_{-\infty}^{\infty} P(Z, t \mid X) \int_{-\infty}^{\infty} P(Y, \Delta t \mid Z) \left(h(Y) - h(Z) \right) dY dZ \right]. \quad [8]$$

Taylor expanding $h(Y)$ about Z gives

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\int_{-\infty}^{\infty} P(Z, t \mid X) \int_{-\infty}^{\infty} P(Y, \Delta t \mid Z) \sum_{n=1}^{\infty} h^{(n)}(Z) \frac{(Y - Z)^n}{n!} dY dZ \right]. \quad [9]$$

Defining the jump moments as

$$D^{(n)}(Z) = \frac{1}{n!} \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty} (Y - Z)^n P(Y, \Delta t \mid Z)dY, \quad [10]$$

it follows that

$$\int_{-\infty}^{\infty} h(Y) \frac{\partial P(Y,t \mid X)}{\partial t} dY = \int_{-\infty}^{\infty} P(Z,t \mid X) \sum_{n=1}^{\infty} D^{(n)}(Z) h^{(n)}(Z) dZ. \qquad [11]$$

Integrating each term on the right side of Eq. **11** by parts n times and using the assumptions on h, after moving terms to the left hand side, it follows that

$$\int_{-\infty}^{\infty} h(Z) \left(\frac{\partial P(Z,t \mid X)}{\partial t} - \sum_{n=1}^{\infty} \left(-\frac{\partial}{\partial Z} \right)^n \left[D^{(n)}(Z) P(Z,t \mid X) \right] \right) dZ = 0. \qquad [12]$$

Now, because h is an arbitrary function, it is necessary that

$$\frac{\partial P(Z,t \mid X)}{\partial t} = \sum_{n=1}^{\infty} \left(-\frac{\partial}{\partial Z} \right)^n \left[D^{(n)}(Z) P(Z,t \mid X) \right]. \qquad [13]$$

We define the probability distribution function $P(X,t)$ of $X(t)$ as the solution of Eq. **13** with initial condition given by a δ-distribution at X_0 at $t = 0$. In this case, $P(X,t) \equiv P(X,t \mid X_0, 0)$ and we may write Eq. **13** as

$$\frac{\partial P(X,t)}{\partial t} = \sum_{n=1}^{\infty} \left(-\frac{\partial}{\partial X} \right)^n \left[D^{(n)}(X) P(X,t) \right], \qquad [14]$$

with

$$D^{(n)}(X_0) = \frac{1}{n!} \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\langle [X(t+\Delta t) - X(t)]^n \right\rangle \Big|_{t=0}, \qquad [15]$$

which is commonly called the Kramers-Moyal expansion. Now, if we assume $D^{(n)}(X) = 0$ for $n > 2$, then we have the Fokker-Planck equation:

$$\frac{\partial P(X,t)}{\partial t} = -\frac{\partial}{\partial X} \left[V(X) P(X,t) \right] + \frac{\partial^2}{\partial X^2} \left[D(X) P(X,t) \right], \qquad [16]$$

where, $V(X) \equiv D^{(1)}(X)$ is the drift coefficient and $D(X) \equiv D^{(2)}(X) > 0$ is the diffusion coefficient, which can be written as

$$V(X_0) = \left. \frac{\partial \langle X(t; X_0) \rangle}{\partial t} \right|_{t=0}, \qquad D(X_0) = \left. \frac{1}{2} \frac{\partial \sigma^2(t; X_0)}{\partial t} \right|_{t=0}, \qquad [17]$$

where angular brackets denote ensemble averaging, σ^2 denotes the variance of X, and $X(t; X_0)$ denotes a realization with $X(0) = X_0$. Any stochastic process $X(t)$ whose probability distribution function satisfies the Fokker-Planck equation is known mathematically as a diffusion process (1).

References

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